# PEIRCE, TURING AND HILBERT: A SKETCH OF PRAGMATISM VS. FORMALISM

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David Hilbert (1862–1943), the leading German mathematician of his generation, expressed through what he termed "my proof theory" the conviction that all questions in mathematics could be answered by algorithmic means.[1] The Hilbert program of formalism, as it came to be known, counted among the items in its charter the belief that the truth or falsehood of any given mathematical statement could be obtained by some mechanical implementation of fixed rules. The idea of such a definite method employing fixed rules explicitly required each group or listing of such rules or steps to be of *finite length.* Each and every mathematical issue was seen as decidable according to some such definite method. This question of decidability, or the *Entscheidungsproblem*, in its German formulation, was a major component of the Hilbert program, a program which sought to delineate all of mathematics in terms of strictly formal properties. From this point of view, the values of given equational symbols in and of themselves are irrelevant: the patterns that occur are what determine meaning. Addressing this issue of formalism for geometry, Hilbert asserted as early as 1891: "It must be possible to replace in all geometric statements the words *point*, *line*, *plane* by *table*, *chair*, *beer-mug*" (Reid 1970: 264). This being the case, the powers of empirical observation involved in using points, lines, and planes in visually observable geometric constructions are critically diminished in value, if not rendered entirely superfluous. What matters, instead, is the formal consistency and integrity of the system itself. For Hilbert, then, the statement "table is to chair as chair is to beer-mug" was *formally* equivalent to "point is to line as line is to plane."

This type of anticipated formal equivalence relies explicitly on the principle of *axiomatization*. The primacy of axioms as components of formalized systems of geometry had been in mathematical currency since antiquity:

> Ever since Euclid, axiomatizing a theory has meant presenting it by singling out certain propositions and deducing further ones from them; if the presentation is complete, it should be the case that *all* statements which could be asserted in the theory are thus deducible. (Parsons 1967)[2]

Thus, a completely axiomatized system should provide for an investigator to deduce each and every true statement within it. Hilbert entertained no doubts that his own efforts toward an exhaustive formalization of mathematics would not only prove successful in this regard, but that such efforts towards a thoroughgoing axiomatization of the subject could be accomplished with actual *ease*: "Hilbert . . . thought of his programme as one of *tidying up loose ends*" (Hodges 1983: 93, emphasis added). Thus, in 1899 Hilbert went beyond his table, chair, and beer-mug speculations of 1891, constructing an axiomatization of Euclidean geometry that did not rely on references to concrete, visually observable examples in the physical world. With this important development Hilbert had accomplished a major step toward separating abstract, formalized aspects of mathematics from possible empirically derived origins and applications. This success in the construction of his "formula game" committed Hilbert to what was clearly an anti-experimental theory of knowledge:

> For this formula game is carried out according to certain definite rules, in which the *technique of our thinking* is expressed. These rules form a closed system that can be discovered and definitively stated. The fundamental idea of my proof theory is none other than to describe the activity of our understanding, to make a protocol of the rules according to which our thinking actually proceeds. . . . If any totality of observations and phenomena deserves to be made the object of serious and thorough investigation, it is this one—since, after all, it is a part of the task of science to liberate us from arbitrariness, sentiment and habit and to protect us from the subjectivism that [has] already made itself felt. (Reid, 1970: 185–86).



Such a closed system, one which operates according to a protocol of rules that can be definitively stated is, in its formalized employment of axioms and their treatment by algorithmic means, what I term a "mechanicalistic" or anti-pragmatic epistemology: it abjures considerations of those same *conceivable* effects that are central to Peirce's pragmatic maxim.[3]

Our story resumes with Hilbert's address to the Second International Congress of Mathematicians held in Paris in 1900. His manuscript contained a list of twenty-three questions whose investigation he believed would help define the future course of mathematics.[4] Here Hilbert's intended axiomatization of the whole of mathematics attained a new sophistication by way of specificity. The last of his questions dealt with devising algorithmic means for, as Hopcroft put it, "establishing the truth or falsity of any statement in [the] language of formal logic called the predicate calculus" (1984: 86). Hilbert believed that the answer to this question must necessarily be an affirmative one, for not to be able to employ a strictly algorithmic, deductive apparatus in the establishment of all statements in that calculus would jeopardize the security of his overall system. In 1904, however, his public declarations on foundationalism ceased, not to be resumed until his address to the Swiss Mathematical Society in Zurich during 1917. Here he announced four problems for the foundations issue. The last of these concerned the decidability of a mathematical question by a finite procedure. As Reid suggests, the lecture might as well have been named "In praise of the axiomatic method" (1970: 151). One might say, in view of the demand for a finite, mechanical procedure, it should have been titled "In praise of the *algorithmic* or mechanicalistic method." Another eleven years were to elapse before Hilbert stated his aim for the algorithmic solvability of mathematical questions in the form which is of interest to us here.

He chose a most conspicuous venue in which to state this program, namely the 1928 International Congress of Mathematicians held in Bologna. This was the first set of international meetings to which the Germans had been invited since World War I:

> At that 1928 congress, Hilbert made his questions